

# On continuity of quantum Fisher information

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We prove an *extended* continuity property for the quantum Fisher information (QFI) and the symmetric logarithmic derivative (SLD), which are general and apply to any case whether associated metrology is closed-system or open system. These properties imply that the QFI and the SLD do not seem to be continuous functions of density matrices; they, however, have this property that for two close density matrices with close first derivatives both the QFIs and the SLDs would be respectively close too. In some special cases these properties reduce to continuity. As an interesting and important result, we establish a bound on the QFI of a quantum state in terms of its manybody entanglement measured by the geometric measure of entanglement. We show that, under some conditions, nonvanishing entanglement seems necessary for a quantum metrology scenario to exhibit quantum-enhanced regime of accuracy. Interestingly, most quantum states chosen randomly from Hilbert space of a manybody system of identical probe systems appear to be sufficiently good to feature such enhanced metrology.

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## I. INTRODUCTION

Estimation of unknown parameters of a system is an essential task for almost all branches of science and technology. Evidently almost any estimation would entail errors due to various factors such as imperfection of measurement devices or natural stochasticity of the event in question. As a result, estimated values are usually inaccurate. It is of fundamental and practical importance to see what optimal accuracy laws of physics allow in principle. This question of fundamental attainable accuracy in metrology can be addressed by the Crámer-Rao bound [1],

$$\delta x \geq (M \mathcal{F}_x^{(C)})^{-1/2}, \quad (1)$$

where ‘ $x$ ’ represents the unknown parameter of interest,  $\delta x$  is the associated estimation error,  $M$  is the number of independent repetitions of the estimation protocol, and measurements are performed on an  $N$ -copy ‘probe’ system. Here the key concept is the (classical) Fisher information,  $\mathcal{F}_x^{(C)}(\{p\})$ , defined as

$$\mathcal{F}_x^{(C)}(\{p\}) = \int_{\mathcal{D}_{x'}} dx' (\partial_x p(x'|x))^2 / p(x'|x), \quad (2)$$

where  $p(x'|x)$  is the conditional probability for obtaining the value  $x'$  given that the exact value of the parameter is  $x$ , and  $\mathcal{D}_{x'}$  is the domain of admissible  $x'$ . One can see that  $\mathcal{F}^{(C)}$  scales as  $O(N)$ , the shot-noise limit [1].

In quantum metrology, a measurement scenario is described by a set of positive operators  $\{\Pi_{x'}\}$  which have the completeness property  $\int_{\mathcal{D}_{x'}} dx' \Pi_{x'} = \mathbb{1}$ . If  $\varrho$  (shorthand for  $\varrho(x)$ ) denotes the (instantaneous) state of the system to be measured, then the probability  $p(x'|x)$  is given by the Born rule  $p(x'|x) = \text{Tr}[\varrho \Pi_{x'}]$ . Here an important quantity is the symmetric logarithmic derivative (SLD), which is a Hermitian operator  $L_\varrho$  defined as [2]

$$\partial_x \varrho = (L_\varrho \varrho + \varrho L_\varrho) / 2. \quad (3)$$

The SLD has the following integral representation [3]:

$$L_\varrho = \int_0^\infty ds e^{-s\varrho} \partial_x \varrho e^{-s\varrho}. \quad (4)$$

Optimizing the Fisher information—attributed to the probabilities obtained through measurements in a quantum metrology scenario—over all measurements yields the quantum Fisher information (QFI)

$$\mathcal{F}_x^{(Q)}(\varrho) = \text{Tr}[\varrho L_\varrho^2]. \quad (5)$$

Thus the quantum Crámer-Rao bound

$$\delta x \geq (M \mathcal{F}^{(Q)})^{-1/2} \quad (6)$$

gives the achievable minimum estimation error [1, 3], where we have used the lighter notation  $\mathcal{F}^{(Q)}$  for the QFI associated with ‘ $x$ .’

Similarly to the classical case, in Eq. (5), the state  $\rho$  denotes the state of the probe system, usually comprised of  $N$  systems (each of which with Hilbert space  $\mathcal{H}$ , hence  $\rho \in \mathcal{S}(\mathcal{H}^{\otimes N})$ , where  $\mathcal{S}$  is the linear space of linear operators on  $\mathcal{H}^{\otimes N}$ ), on which a measurement strategy is applied. Bearing this in mind, however, it can be expected that existence of quantum features may enhance metrology. Particularly, it has been shown that a quantum mechanical enhancement in the form of  $O(N^2)$  scaling (the Heisenberg limit) can be achieved by employing manybody quantum correlations (e.g., entanglement) [4, 5], or manybody interactions [6], and nonlinearities [7]. In fact, most of the existing literature on quantum metrology is focused on the scaling of the QFI with the probe size, under various conditions, in closed and open (i.e., noisy or dissipative) metrology scenarios [8–11]. For a review of this subject, see, e.g., Refs. [12, 13].

Despite this vast understanding, not much is yet known about specific properties of the QFI [2, 13, 14]. For example, among important properties to investigate is *convexity* [15–17], which just recently has been shown to hold for the QFI in the following *extended* form [18]:

$$\mathcal{F}^{(Q)}\left(\sum_a p_a \rho_a\right) \leq \mathcal{F}^{(c)}(\{p_a\}) + \sum_a p_a \mathcal{F}^{(Q)}(\rho_a). \quad (7)$$

Another crucial property to study is *continuity*, in the sense that whether two ‘close’ states (defined in some specific sense) have close QFIs. Such a property has already been shown to hold for, e.g., the von Neumann entropy [19, 20] and some (entropy-based) entanglement measures [21–23]. For the QFI, however, this property up until now has not been studied in a general manner. This paper is to bridge this gap. Here we derive an *extended* continuity relation for the QFI. This continuity is general in that it is independent of the underlying dynamics for the probe system or how the unknown parameter enters in the probe state. As an application, we relate this continuity with the manybody entanglement of the probe system. This relation enables us to see why entanglement may enhance estimation accuracy to a sub-shot noise or Heisenberg regime.

The structure of this paper is as follows. In Sec. II we provide some preliminaries for our derivation. In Sec. III we lay out our main results and prove them. Section IV deals with establishing a relation between the QFI and entanglement of probe states. We conclude the paper in Sec. V.

## II. PRELIMINARIES

In this section, we establish the preliminaries required for proving our main results.

We begin by reminding the definitions of the  $p$ -norm ( $p \in [1, \infty]$ ) of linear operators. For a linear operator  $A$  acting on a linear space, it is defined that  $\|A\|_p = (\text{Tr}[|A|^p])^{1/p}$ , where  $|A| = \sqrt{A^\dagger A}$  [24]. This norm has numerous appealing properties [24–26]. One particular property useful for our purpose in this paper is the following duality between the 1-norm (trace norm) and the  $\infty$ -norm (the standard sup-operator norm) [26, 27]:

$$\|A\|_1 = \sup_{B \neq 0} \frac{|\text{Tr}[B^\dagger A]|}{\|B\|_\infty}, \quad (8)$$

from whence we obtain this useful inequality

$$|\text{Tr}[AB]| \leq \|A\|_1 \|B\|_\infty \leq \|A\|_1 \|B\|_1. \quad (9)$$

The last inequality is a special case of the property

$$\|A\|_q \leq \|A\|_p, \quad 1 \leq p \leq q \leq \infty. \quad (10)$$

Another useful property is the submultiplicativity in the form

$$\|AB\|_1 \leq \|A\|_\infty \|B\|_1, \quad \|A\|_1 \|B\|_\infty. \quad (11)$$

One can use the definition of the  $p$ -norm to define an induced norm for linear *maps*. A linear map  $\mathcal{E}$  acting on linear operators is defined through  $\mathcal{E}[X] = \sum_i A_i X B_i$ , for some set of operators  $\{A_i\}$  and  $\{B_i\}$ , where  $X$  is an arbitrary linear operator (note that all operators here act on the same linear space) [24, 25]. One can define [26, 28]

$$\|\mathcal{E}\|_p = \sup_{X \neq 0} \frac{\|\mathcal{E}[X]\|_p}{\|X\|_p}. \quad (12)$$

If  $\mathcal{E}$  is a positive, trace-preserving map (where  $\mathcal{E}[X] \geq 0$  if  $X \geq 0$  and  $\text{Tr}[\mathcal{E}[X]] = \text{Tr}[X]$ ), it has been shown that  $\|\mathcal{E}\|_1 \leq 1$  [28]. Thus for quantum maps [20] we have  $\|\mathcal{E}\|_1 \leq 1$ .

For any pair of general linear maps  $\mathcal{E}_1$  and  $\mathcal{E}_2$  one can see that

$$\|\mathcal{E}_1[X_1] - \mathcal{E}_2[X_2]\|_p = \|(\mathcal{E}_1 - \mathcal{E}_2)[X_2] + \mathcal{E}_1[X_1 - X_2]\|_p \quad (13)$$

$$\leq \|\mathcal{E}_1 - \mathcal{E}_2\|_p \|X_2\|_p + \|\mathcal{E}_1\|_p \|X_1 - X_2\|_p. \quad (14)$$

In addition, when  $\mathcal{E}_a[X] = A_a X A_a^\dagger$  ( $a = 1, 2$ ), a straightforward calculation shows that

$$\|\mathcal{E}_1[X] - \mathcal{E}_2[X]\|_p = \|A_1 X A_1^\dagger - A_2 X A_2^\dagger\|_p \quad (15)$$

$$\begin{aligned} &= \|(A_1 - A_2)X(A_1^\dagger - A_2^\dagger) - 2A_2 X A_2^\dagger + A_2 X A_1^\dagger + A_1 X A_2^\dagger\|_p \\ &= \|(A_1 - A_2)X(A_1^\dagger - A_2^\dagger) - A_2 X(A_1^\dagger - A_2^\dagger) + (A_1 - A_2)X A_2^\dagger\|_p \\ &\leq \|(A_1 - A_2)X(A_1^\dagger - A_2^\dagger)\|_p + \|A_2 X(A_1^\dagger - A_2^\dagger)\|_p + \|(A_1 - A_2)X A_2^\dagger\|_p \\ &\leq \|X\|_p \|A_1 - A_2\|_p (\|A_1 - A_2\|_p + 2\|A_2\|_p), \end{aligned} \quad (16)$$

where we have used  $\|O^\dagger\|_p = \|O\|_p$ , which is valid for any bounded linear operator [24]. Thus from Eq. (12) it is evident that

$$\|\mathcal{E}_1 - \mathcal{E}_2\|_p \leq \|A_1 - A_2\|_p (\|A_1 - A_2\|_p + 2\|A_2\|_p). \quad (17)$$

Inserting this relation into Eq. (14) yields

$$\|\mathcal{E}_1[X_1] - \mathcal{E}_2[X_2]\|_p \leq \|A_1 - A_2\|_p (\|A_1 - A_2\|_p + 2\|A_2\|_p) \|X_2\|_p + \|A_1\|_p^2 \|X_1 - X_2\|_p. \quad (18)$$

Another identity which will be important for our analysis is as follows:

$$e^A - e^B = \int_0^1 d\tau e^{\tau A} (A - B) e^{(1-\tau)B}, \quad (19)$$

for any pair of linear operators defined on a given linear space. To prove this, we choose  $V(\tau) = e^{\tau A} e^{-\tau B}$ , and use  $V(1) - V(0) = \int_0^1 d\tau dV(\tau)/d\tau$ . An immediate consequence of this formula is

$$\|e^A - e^B\|_p \leq \|A - B\|_p \int_0^1 d\tau \|e^{\tau A}\|_p \|e^{(1-\tau)B}\|_p. \quad (20)$$

A useful special case is when  $A = -s\varrho$ ,  $B = -s\sigma$ ,  $s \geq 0$ , in which  $\varrho$  and  $\sigma$  are two quantum states of a given system. We first note that

$$\|e^{-s\tau\varrho}\|_\infty = e^{-s\tau\lambda_{\min}(\varrho)}, \quad (21)$$

where  $\lambda_{\min}(\varrho)$  is the smallest eigenvalue of  $\varrho$  (and similarly for  $\|e^{-s\tau\sigma}\|_\infty$ ). Using this relation, we can calculate the integral in Eq. (20), from whence

$$\|e^{-s\varrho} - e^{-s\sigma}\|_\infty \leq \frac{e^{-s\lambda_{\min}(\sigma)} - e^{-s\lambda_{\min}(\varrho)}}{\lambda_{\min}(\varrho) - \lambda_{\min}(\sigma)} \|\varrho - \sigma\|_1. \quad (22)$$

Applying a similar method to Eq. (4) gives an upper bound for  $\|L_\varrho\|_\infty$  as follows:

$$\|L_\varrho\|_\infty \leq \|\partial_x \varrho\|_\infty / \lambda_{\min}(\varrho). \quad (23)$$

The next relation we need to remind is the definition of the fidelity of two quantum states  $\varrho$  and  $\sigma$ ,  $F(\varrho, \sigma) = \text{Tr} \sqrt{\sqrt{\varrho} \sigma \sqrt{\varrho}}$ , which is always  $0 \leq F(\varrho, \sigma) \leq 1$ , and is a measure of closeness of the two quantum states. There exists a useful relation between the fidelity and the 1-norm distance of two density matrices as follows [2, 20]:

$$1 - F(\varrho, \sigma) \leq \frac{1}{2} \|\varrho - \sigma\|_1 \leq \sqrt{1 - F^2(\varrho, \sigma)}. \quad (24)$$

It is evident that  $0 \leq (1/2)\|\varrho - \sigma\|_1 \leq 1$ .

As explained in Sec. I, understanding how the QFI scales quantum mechanically as  $O(N^{1+\delta})$ , where  $\delta > 0$ , with the probe size is an important question. To answer this question, one can, for example, keep track of manybody entanglement present in

the  $N$ -probe system. We thus need to have a plausible measure of manybody entanglement. One particularly useful measure, which has been employed widely in the literature, is the geometric measure of entanglement, defined as

$$E_G(\varrho) = 1 - \max_{\sigma \in \mathcal{S}} F^2(\varrho, \sigma), \quad (25)$$

where

$$\mathcal{S} = \left\{ \sum_i q_i |\varphi_i^{(1)}\rangle \langle \varphi_i^{(1)}| \otimes \dots \otimes |\varphi_i^{(N)}\rangle \langle \varphi_i^{(N)}| ; |\varphi_i^{(j)}\rangle \in \mathcal{H} \forall j, 0 \leq q_i \leq 1, \sum_i q_i = 1 \right\}. \quad (26)$$

That is, the maximization is taken over all separable states which form the convex roof of fully product states [29]. It is obvious that  $0 \leq E_G(\varrho) \leq 1$ . This measure can be related to the distance of the probe state to the set of the convex roof of fully product states in  $\mathcal{H}^{\otimes N}$ . Specifically, Eq. (24) yields

$$\min_{\sigma \in \mathcal{S}} \|\varrho - \sigma\|_1 \leq 2 \sqrt{1 - \max_{\sigma \in \mathcal{S}} F^2(\varrho, \sigma)} = 2 \sqrt{E_G(\varrho)}. \quad (27)$$

We shall use this formula later to relate the QFI to the entanglement of the probe state.

### III. EXTENDED CONTINUITY OF THE QFI

From the definition of the QFI (5) for two parameter-dependent states  $\varrho$  and  $\sigma$ , we have

$$\begin{aligned} \mathcal{F}^{(Q)}[\varrho] - \mathcal{F}^{(Q)}[\sigma] &= \text{Tr}[L_\varrho \varrho L_\varrho] - \text{Tr}[L_\sigma \sigma L_\sigma] \\ &\stackrel{(9)}{\leq} \|L_\varrho \varrho L_\varrho - L_\sigma \sigma L_\sigma\|_1. \end{aligned} \quad (28)$$

If we replace in Eq. (18)  $A_1 = L_\varrho$ ,  $A_2 = L_\sigma$ ,  $X_1 = \varrho$ , and  $X_2 = \sigma$ , we have

$$|\mathcal{F}^{(Q)}[\varrho] - \mathcal{F}^{(Q)}[\sigma]| \leq \|L_\varrho - L_\sigma\|_\infty (\|L_\varrho - L_\sigma\|_\infty + 2\|L_\sigma\|_\infty) + \|L_\varrho\|_\infty^2 \|\varrho - \sigma\|_1, \quad (29)$$

where we have used the fact that for quantum states we have  $\|\sigma\|_1 = \text{Tr}[\sigma] = 1$ , and that  $\|X\|_\infty \leq \|X\|_1$ . Equation (29) indicates that we still need to calculate  $\|L_\varrho - L_\sigma\|_\infty$  in terms more of primitive quantities (such as  $\varrho$ ,  $\sigma$ , and perhaps their derivatives).

Now we want to show that in order for  $L_\varrho$  to be close to  $L_\sigma$ , it is required that not only  $\varrho$  becomes close to  $\sigma$ , but also  $\partial_x \varrho$  becomes close to  $\partial_x \sigma$ . To this end, we start from the integral form of the SLD (4),

$$L_\varrho - L_\sigma = 2 \int_0^\infty ds (e^{-s\varrho} \partial_x \varrho e^{-s\varrho} - e^{-s\sigma} \partial_x \sigma e^{-s\sigma}) \quad (30)$$

whence we obtain

$$\|L_\varrho - L_\sigma\|_\infty \leq 2 \int_0^\infty ds \|e^{-s\varrho} \partial_x \varrho e^{-s\varrho} - e^{-s\sigma} \partial_x \sigma e^{-s\sigma}\|_\infty, \quad (31)$$

Once again, we employ Eq. (18) with  $A_1 = e^{-s\varrho}$ ,  $A_2 = e^{-s\sigma}$ ,  $X_1 = \partial_x \varrho$ , and  $X_2 = \partial_x \sigma$ , which yields

$$\|L_\varrho - L_\sigma\|_\infty \leq 2 \int_0^\infty ds \left[ \|e^{-s\varrho} - e^{-s\sigma}\|_\infty (\|e^{-s\varrho} - e^{-s\sigma}\|_\infty + 2\|e^{-s\sigma}\|_\infty) \|\partial_x \varrho - \partial_x \sigma\|_\infty + \|e^{-s\varrho}\|_\infty^2 \|\partial_x \varrho - \partial_x \sigma\|_\infty \right]. \quad (32)$$

Now if we use Eqs. (21) and (22), the integrals of the right-hand side of Eq. (32) can be calculated leading to

$$\|L_\varrho - L_\sigma\|_\infty \leq \frac{\|\partial_x \varrho\|_\infty [2\lambda_{\min}(\varrho) + \|\varrho - \sigma\|_1]}{\lambda_{\min}(\varrho) \lambda_{\min}(\sigma) [\lambda_{\min}(\varrho) + \lambda_{\min}(\sigma)]} \|\varrho - \sigma\|_1 + \frac{1}{\lambda_{\min}(\varrho)} \|\partial_x \varrho - \partial_x \sigma\|_\infty, \quad (33)$$

or, after some further algebra, to the looser version

$$\|L_\varrho - L_\sigma\|_\infty \leq (1/2) \nu \|\partial_x \varrho\|_\infty [2 + \|\varrho - \sigma\|_1] \|\varrho - \sigma\|_1 + \frac{1}{\lambda_{\min}(\varrho)} \|\partial_x \varrho - \partial_x \sigma\|_\infty, \quad (34)$$

where  $\nu = \lambda_{\min}^{-4}(\varrho)\lambda_{\min}^{-4}(\sigma)$ . This relation is interesting and important per se, and it establishes an *extended* continuity property for the SLD. In particular, this relation implies that for two SLDs to be close, not only the two density matrices should be close, but also their first derivatives should be close. This seems natural if we note that the SLD (3) requires both the density matrix and its first derivative.

Due to Eq. (29), the latter extended continuity of the SLD carries over to the QFI too. More explicitly, after some algebra we obtain

$$|\mathcal{F}^{(Q)}[\varrho] - \mathcal{F}^{(Q)}[\sigma]| \leq \sum_{m=1}^4 f_m \|\varrho - \sigma\|_1^m + \sum_{n=1}^2 g_n \|\partial_x \varrho - \partial_x \sigma\|_\infty^n, \quad (35)$$

where

$$f_1 = 2\nu [\|\partial_x \sigma\|_\infty \|\partial_x \varrho - \partial_x \sigma\|_\infty + \|\partial_x \sigma\|_\infty^2 + 2\|\partial_x \varrho\|_\infty^2], \quad (36)$$

$$f_2 = \nu \|\partial_x \sigma\|_\infty [\|\partial_x \sigma\|_\infty + \|\partial_x \sigma\|_\infty + \|\partial_x \varrho - \partial_x \sigma\|_\infty], \quad (37)$$

$$f_3 = (1/2)\nu \|\partial_x \sigma\|_\infty^2, \quad (38)$$

$$f_4 = (1/4)\nu \|\partial_x \sigma\|_\infty^2, \quad (39)$$

$$g_1 = 2\nu \|\partial_x \sigma\|_\infty, \quad (40)$$

$$g_2 = \nu. \quad (41)$$

*Remark 1.*—As a caveat we remark that the bounds and inequalities we derive in this paper are not necessarily tight.

#### IV. ROLE OF ENTANGLEMENT IN QUANTUM METROLOGY

Here we discuss an important utility of our extended continuity relations. It is interesting that, in light of Eq. (27), one can also recast the extended continuity relation (29)—or its detailed version (35)—in terms of the geometric measure of entanglement of the state  $\varrho$ , if we take  $\sigma \in \mathcal{S}$  and perform an optimization as in Eq. (27). This leads to

$$\mathcal{F}^{(Q)}(\varrho) \leq \mathcal{F}^{(Q)}(\sigma_*) + \sum_{m=1}^4 2^m f_m E_G^{m/2}(\varrho) + \sum_{n=1}^2 g_n \|\partial_x \varrho - \partial_x \sigma\|_\infty^n \quad (42)$$

$$\leq \mathcal{F}^{(Q)}(\sigma_*) + (16 \sum_{m=1}^4 f_m) \sqrt{E_G(\varrho)} + \sum_{n=1}^2 g_n \|\partial_x \varrho - \partial_x \sigma\|_\infty^n, \quad (43)$$

where  $\sigma_* \in \mathcal{S}$  is the separable state which saturates the maximum in the right-hand side of Eq. (27), or equivalently the state which minimizes  $\|\varrho - \sigma\|_1$  for a given  $\varrho$ .

To further illustrate the utility of Eq. (42) [or Eq. (43)], we consider the special (and prototypical) case of a unitary single-parameter (‘phase’) estimation scenario, in which the dependence on the parameter  $x$  comes through

$$\varrho = e^{-ixH} \varrho_0 e^{ixH}, \quad (44)$$

where  $H$  is a Hamiltonian acting on the  $N$ -probe system, and  $\varrho_0$  is its initial state [4, 31]. In this case, we have

$$\partial_x \varrho = -i[H, \varrho], \quad (45)$$

and similarly for  $\sigma$ . Hence

$$\|\partial_x \varrho\|_\infty, \|\partial_x \sigma\|_\infty \leq 2\|H\|_\infty, \quad (46)$$

$$\|\partial_x \varrho - \partial_x \sigma\|_\infty \leq 2\|H\|_\infty \|\varrho - \sigma\|_1 = 2\|H\|_\infty \|\varrho_0 - \sigma_0\|_1. \quad (47)$$

From these relations and the extended continuity of the SLD (32), it is seen that in this special case the SLD is indeed continuous because

$$\|L_\varrho - L_\sigma\|_\infty \leq 2\|H\|_\infty \|\varrho - \sigma\|_1 \left( \frac{\lambda_{\min}(\sigma)[\lambda_{\min}(\varrho) + \lambda_{\min}(\sigma)] + 2\lambda_{\min}(\varrho) + \|\varrho - \sigma\|_1}{\lambda_{\min}(\varrho)\lambda_{\min}(\sigma)[\lambda_{\min}(\varrho) + \lambda_{\min}(\sigma)]} \right), \quad (48)$$

or

$$\|L_\varrho - L_\sigma\|_\infty \leq \nu \|H\|_\infty \|\varrho - \sigma\|_1 [4 + \|\varrho - \sigma\|_1], \quad (49)$$

where in the last equation we have used  $\lambda_{\min}(\varrho), \lambda_{\min}(\sigma) \leq 1$ . From Eq. (29) [or Eq. (35)], this continuity of the SLD carries over to the QFI too.

*Remark 2.*—Note that due to the unitarity of the dynamics and the unitary invariance of the 1-norm [24], in the right-hand side of the inequalities for the unitary evolution case we can replace  $\varrho$  and  $\sigma$  with  $\varrho_0$  and  $\sigma_0$ , respectively.

If we choose  $\sigma = \sigma_*$ , the separable state described below Eq. (43), after some straightforward algebra we can rewrite Eq. (49) in terms of  $E_G(\varrho)$  as

$$\|L_\varrho - L_{\sigma_*}\|_\infty \leq 12\nu_* \|H\|_\infty \sqrt{E_G(\varrho)}. \quad (50)$$

Inserting this relation in Eq. (29) gives

$$\mathcal{F}^{(Q)}(\varrho) \leq \mathcal{F}^{(Q)}(\sigma_*) + 208\nu^2 \|H\|_\infty^2 \sqrt{E_G(\varrho)}. \quad (51)$$

By using the extended convexity relation (7) we can also derive an upper bound for  $\mathcal{F}^{(Q)}(\sigma_*)$ . Consider that  $\sigma_* = \sum_a q_a |\varphi_a^{(1)}\rangle\langle\varphi_a^{(1)}| \otimes \dots \otimes |\varphi_a^{(N)}\rangle\langle\varphi_a^{(N)}|$ , where both  $q_a$ s and  $|\varphi_a^{(j)}\rangle$ s may have the parameter (x) dependence. Thus

$$\begin{aligned} \mathcal{F}^{(Q)}(\sigma_*) &\leq \mathcal{F}^{(C)}(\{q_a\}) + \sum_a q_a \mathcal{F}^{(Q)}(\otimes_{j=1}^N |\varphi_a^{(j)}\rangle\langle\varphi_a^{(j)}|) \\ &\leq \alpha_C N + \sum_a q_a \sum_{j=1}^N \mathcal{F}^{(Q)}(|\varphi_a^{(j)}\rangle\langle\varphi_a^{(j)}|) \\ &= (\alpha_C + \alpha_Q)N, \end{aligned} \quad (52)$$

where  $\mathcal{F}^{(C)}(\{q_a\}) = \alpha_C N$ ,  $\alpha_Q = \max_a \max_j \mathcal{F}^{(Q)}(|\varphi_a^{(j)}\rangle\langle\varphi_a^{(j)}|)$ , and we have used the fact that the QFI has the additivity property for tensor product states [13, 18]. As a result, Eq. (51) reduces to

$$\mathcal{F}^{(Q)}(\varrho) \leq \alpha_{CQ} N + 208\nu^2 \|H\|_\infty^2 \sqrt{E_G(\varrho)}, \quad (53)$$

where  $\alpha_{CQ} = \alpha_C + \alpha_Q$ . We can go even further by considering that the Hamiltonian  $H$  acting on the  $N$ -probe system can be  $k$ -local [6], i.e.,  $H = \sum_{\{j_1, \dots, j_k\}} H_{j_1, \dots, j_k}^{(k)}$ . Although  $k \geq 1$ , in physical systems it is limited to  $k = 1$  or at most 2 (unless engineered artificially). Hence by using the fact that  $\|H\|_\infty \leq \binom{N}{k} \max_{\{j_1, \dots, j_k\}} \|H_{j_1, \dots, j_k}^{(k)}\|_\infty$ , we can further simplify Eq. (53) as

$$\mathcal{F}^{(Q)}(\varrho) \leq \alpha_{CQ} N + \alpha_{QQ} \binom{N}{k}^2 \nu^2 \sqrt{E_G(\varrho)}, \quad (54)$$

$$\lesssim \alpha_{CQ} N + \alpha_{QQ} N^{2k} \nu^2 \sqrt{E_G(\varrho)} / k!, \quad (55)$$

where  $\alpha_{QQ} = 208 [\max_{\{j_1, \dots, j_k\}} \|H_{j_1, \dots, j_k}^{(k)}\|_\infty]^2$ . This is an interesting relation which shows for a quantum-enhanced scaling  $O(N^{1+\eta})$  (with  $\eta > 0$ ) it is important to have both  $\nu$  and the geometric entanglement  $E_G(\varrho)$  nonvanishing. This relation can be used to identify when a quantum metrology features classical or quantum scalings. A precursor for such “classical-quantum” tradeoff relations has already been studied in Ref. [18].

Both  $\nu$  and  $E_G(\varrho)$  factors may also scale with  $N$  as  $O(N^{-\kappa})$ , with some positive  $\kappa$ , and thus can change the scaling of the second term in the right-hand side of Eq. (55). This is a caveat that even in the presence of entanglement there might still be cases where no quantum enhancement is observed [31]. A general discussion of such dependence would be a formidable task beyond the scope of this paper. However, one still can draw interesting conclusions about some special cases. For instance, let us see how a (uniformly) randomly selected state from  $\mathcal{H}^{\otimes N}$  can perform in a metrology task. It has been known that the fraction of states  $|\Psi\rangle$  on the Hilbert space of  $N$  qubits  $\mathcal{H}^{\otimes N}$  (where  $\mathcal{H} = \mathbb{C}^2$  and  $N \geq 11$ ) with  $E_G(|\Psi\rangle) < 1 - 2^{-N+\delta}$  (with  $\delta$  being a logarithmic modification) is smaller than  $e^{-N^2}$  [32]. In other words, *most* quantum states of an  $N$ -qubit probe system are indeed highly entangled in that their geometric measure of entanglement is typically close to 1, and thus may be useful for enhancing metrology (at least as far as the upper bound (55) suggests).

## V. SUMMARY AND CONCLUSIONS

We have proved extended continuity relations for the quantum Fisher information (QFI) and the symmetric logarithmic derivative (SLD). These properties imply that, in general cases, the QFI and the SLD do not seem to be continuous functions of density

matrices. Rather, the QFI and the SLD behave in a way that for two close density matrices with close first derivatives both the QFIs and the SLDs would be respectively close too. We have shown that in some special cases (e.g., in a unitary dynamics), this extended continuity reduces to a continuity property for both the QFI and the SLD. Our continuity relations are general in that they hold irrespective of that a metrology is a closed-system or an open-system (noisy) scenario. As such, our results are powerful and can apply to any metrology. Despite this generality, our proofs have been fairly straightforward, based mostly on operator norm inequalities.

We next have used this extended continuity property of the QFI to establish a general quantitative connection among utility of a quantum probe state for quantum-enhanced metrology, the role of dynamics therein, and quantum correlation content of the probe state in the form of entanglement. In particular, we have demonstrated that—as far as the continuity bound is concerned and under some specific conditions—a nonvanishing entanglement (measured by the geometric measure of entanglement) seems necessary for a quantum metrology scenario to exhibit quantum-enhanced performance. We have also used a typicality property of random quantum states of manybody systems—that typically they are highly entangled—to argue that most probe states appear to be sufficiently good to feature quantum-enhanced metrology.

We anticipate that, given the generality and utility of our extended continuity relations, they can spur numerous applications in quantum metrology and perhaps other areas of quantum information science and technology.

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